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Optimal control of forced cool-down processes

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Abstract

The optimization of forced cooling-down processes is performed using the optimal control theory. Four objective functions are shortly presented and minimized. They include the consummated cooling fluid mass and some dissipation measures, as the entropy generation and variants of the lost available work. The minimum duration for the cooling process with minimum lost available work normally exceeds the minimum duration for the cooling processes with minimum mass of cooling fluid but the reverse is sometimes true. The minimum lost available work does not depend on the initial and final temperatures of the cooled body but just on the duration of the cooling process. The cooling strategies with linear (or slightly non-linear) time dependence of cooled body temperature are based on rather strong non-linear increase in the optimum cooling fluid mass flow rate at the end of the cooling process.

Keywords: Cool-down process; Entropy generation; Lost available work; Optimal control

1. Introduction

An early optimization of forced cooling-down processes was made in [1]. The authors used a variational calculus approach and the objective function to be minimized was the consummated cooling fluid mass. Indeed, the cooling fluid is an expensive commodity because it is proportional to its energy content and to the actual refrigerator power required to produce it. Cooling processes are wildly used in metallurgy, chemical industry and other activities traditionally covered by thermal engineering, but also in electronics (see e.g. [2,3]). The "cool-down" problem is of significant importance for example in cryogenics, where large-scale superconducting windings must first be cooled to liquid-helium temperature before they can be operational [4].

Other objective functions, based on various dissipation measures, could also be envisaged for cooling process analysis. A collection of design techniques related to a dissipation measure is for instance the method of entropy generation minimization, which became a popular tool in thermal engineering in the last 20 years. It consists of a mixture of classical thermodynamics, heat and mass transfer, and fluid mechanics [5]. For pioneering studies see e.g. [6,7]. For recent results see for example [8–10]. A number of good reviews are available, as [1,11,12] to quote a few.

More involved dissipation measures, as for example the lost available work, are equally useful. Indeed, the effective management of the available work is the primary objective of various industrial activities of mechanical, thermal, electrical or chemical nature. In many

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Nomenclature

A	heat transfer surface area [m^2]
C	specific heat of cooled body [J/kg/K]
C', C''	integration constants
c	specific heat of cooling fluid [J/kg/K]
COP	coefficient of performance
F	function defined in Eq. (3.15)
H	Hamiltonian
M	mass of cooled body [kg]
m	mass of cooling fluid [kg]
$\dot{m}$	mass flow rate of cooling fluid [kg/s]
Q	heat flux [W]
R, R'	functions defined in Table 2
S	entropy [J/K]
$\tilde{S}$	dimensionless entropy
$\dot{S}$	entropy rate [W/K]
$\dot{\tilde{S}}$	dimensionless entropy rate
T	temperature, cooled body temperature [K]
t	time [s]
U	heat transfer coefficient [W/m^2/K]
W	work [J]
$\widetilde{W}$	dimensionless work
$\dot{W}$	work rate [W]
$\dot{\tilde{W}}$	dimensionless work rate
z	dimensionless cooled body temperature
y	dimensionless cooling fluid temperature

Greek symbols

μ	dimensionless cooling fluid mass
$\dot{\mu}$	dimensionless cooling fluid mass flow rate
ψ	adjunct function
τ	dimensionless time interval
τ'	dimensionless time (minimum mass)
τ''	dimensionless time (minimum lost work)
ω	dimensionless time

Subscripts and superscripts

0	referring to cooling fluid reservoir
0,1	the number of the adjunct function
$0 \rightarrow t_c$	from start to end of cooling process
(a)	first strategy of minimum lost work
(b)	second strategy of minimum lost work
b → f	from cooled body to cooling fluid
c	cooling process
f,out	cooling fluid, outlet
fin	final
gen	generated
init	initial
in → out	between inlet and outlet
l	lost
opt	optimum
min	minimum
p	constant pressure

situations of practical interest, the minimum of entropy generation during a process is associated with a minimum of the lost available work. However, there are cases where the minimum of the entropy generation and the minimum of the lost available work do not coincide. For a recent study on this subject see [13] where various examples are given, including devices operating from a heat reservoir and solar and geothermal power plants.

The heating/cooling strategies are different from the point of view of their costs and feasibility. Therefore, optimization of heating/cooling processes can yield a variety of answers, depending not only on the objective of the optimization but also on the constraints that define the problem [13]. Designers use now a number of well-developed techniques, which allow optimization of both heating/cooling equipment structure and operation, as for example the method of thermodynamic cycles, the method of thermodynamic potentials and the availability (or second-low) analysis. The cooling process optimization is treated in this paper by using a more advanced method, namely the optimal control theory developed by Pontryagin et al. [14]. The optimal control theory was mainly developed and used in the past for mechanical applications (e.g. for aircraft and spacecraft operation). It is sometimes used in thermal engineering (see e.g. [15,16] and the excellent review [17]).

The objective function used previously in [1] is first considered in this paper. Also, the forced cooling processes is optimized on the basis of three dissipation measures, namely the entropy generation and two other measures associated to the lost available work.

2. Forced cooling processes with minimization of cooling fluid mass

The cooling process early treated in [1] is presented now (Fig. 1). A body of mass M, specific heat C and time-dependent (and space-uniform) temperature $T(t)$ has to be cooled during a given time interval t_c, from the initial temperature T_{init} to the final temperature T_{fin}, by using a cooling fluid of constant isobaric specific heat c_p. The fluid (time-dependent) mass flow rate is denoted

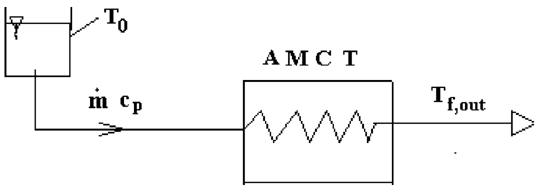

Fig. 1. Cooling process of a body of mass M, specific heat C and temperature T. The mass flow rate of the cooling fluid is $\dot{m}$ while its isobaric specific heat is c_p. The fluid comes from the reservoir at temperature T_0 and leaves the body at temperature $T_{f,out}$. The heat transfer area between body and cooling fluid is A.

$\dot{m}(t)$. The heat transfer surface area between body and fluid is A while the (constant in time) heat transfer coefficient is U. The fluid is provided by a reservoir of constant temperature $T_0(<T_{init})$. While in contact with the body, the fluid has a uniform (time-dependent) temperature $T_{f,out}(t)$. The last hypothesis is well verified for example in case of macro-porous super-conducting structures or in case of large liquid storage tanks well stirred [1].

The heat flux transferred by Newtonian convection from the body to the fluid is denoted $Q_{b\to f}$ and the thermal energy flux received by the fluid while in contact with the body is denoted $Q_{f,in\to out}$. These two fluxes are given by the following simple relationships:

$$Q_{b\to f} = UA\lfloor T(t) - T_{f,out}(t)\rfloor$$
$$Q_{f,in\to out} = \dot{m}c_p\lfloor T_{f,out}(t) - T_0\rfloor \quad (2.1, 2)$$

No energy loss is considered during the heat transfer process. Then, use of Eqs. (2.1), (2.2) and the first law yields

$$MC\frac{dT}{dt} = -Q_{f,in\to out} = -Q_{b\to f} = -UA[T(t) - T_{f,out}(t)] \quad (2.3)$$

The mass $m_{0\to t_c}$ of fluid consummated from the reservoir during the cooling process is given by:

$$m_{0\to t_c} = \int_0^{t_c} \dot{m}(t)\,dt \quad (2.4)$$

Now, the optimization problem is presented. One look about that particular time evolution of the cooling fluid mass flow rate (say $\dot{m}_{opt}(t)$) which makes $m_{0\to t_c}$ given by Eq. (2.4) to be a minimum. The constraints Eqs. (2.1)–(2.3) should also be taken into account.

A dimensionless formulation is convenient. The dimensionless variables are first defined:

$$\omega \equiv \frac{t}{t_c} \quad z(\omega) \equiv \frac{T}{T_0} \quad y(\omega) \equiv \frac{T_{f,out}}{T_0} \quad \dot{\mu}(\omega) \equiv \frac{\dot{m}c_p}{UA} \quad (2.5)$$

Also, dimensionless constants are defined:

$$z_{init} \equiv \frac{T_{init}}{T_0} \quad z_{fin} \equiv \frac{T_{fin}}{T_0} \quad \tau_c \equiv \frac{UAt_c}{MC} \quad (2.6)$$

The following relationships exist for the independent and dependent dimensionless variables:

$$0 \leqslant \omega \leqslant 1 \quad z_{fin} \leqslant z \leqslant z_{init} \quad y(\omega) \leqslant z(\omega) \quad (2.7)$$

Use of Eqs. (2.1)–(2.7) allows to write the relationships giving the time dependence of the dimensionless temperatures z and y

$$\frac{dz}{d\omega} = -\frac{\tau_c\dot{\mu}}{1+\dot{\mu}}(z-1) \quad y = \frac{z+\dot{\mu}}{1+\dot{\mu}} \quad (2.8, 9)$$

The ordinary differential Eq. (2.8) must be solved by using the following boundary conditions:

$$z(\omega = 0) = z_{init} \quad z(\omega = 1) = z_{fin} \quad (2.10)$$

The dimensionless objective function μ is defined by using Eq. (2.4) as follows:

$$\mu \equiv \frac{m_{0\to t_c}c_p}{MC} = \tau_c\int_0^1 \dot{\mu}\,d\omega \quad (2.11)$$

The optimization problem consists now in the minimization of μ given by Eq. (2.11), by taking into account the boundary conditions Eq. (2.10) and the constraint Eq. (2.8).

A good introduction to optimal control theory may be found in several books (see e.g. [18]). The theory is applied here with the dimensionless mass flow rate $\dot{\mu}$ as the control function. Two adjunct functions (say $\psi_0(\omega)$ and $\psi_1(\omega)$) are used. The Eqs. (2.8) and (2.11) allow to define the Hamiltonian H as follows:

$$H \equiv \psi_0(\tau_c\dot{\mu}) + \psi_1\left[-\frac{\tau_c\dot{\mu}}{1+\dot{\mu}}(z-1)\right] \quad (2.12)$$

The values of the unknown function $z(\omega)$ at the end-points of the integration interval (i.e. at $\omega = 0$ and $\omega = 1$) are known (they are given by Eq. (2.10)). According to Pontryagin's theory one ca use in this case $\psi_0 = -1$ (see e.g. [19]). The adjunct function ψ_1 obeys the equation $d\psi_1/d\omega = -\partial H/dz$. Use of Eq. (2.11) yields:

$$\frac{d\psi_1}{d\omega} = \frac{\tau_c\dot{\mu}}{1+\dot{\mu}}\psi_1 \quad (2.13)$$

One eliminates $\dot{\mu}$ between the two ordinary differential Eqs. (2.8) and (2.13). Solving the resulting equation yields:

$$-\psi_1(z-1) = C' = ct \quad (2.14)$$

where C' is an integration constant.

The optimal control function (say $\mu_{opt}(\omega)$) can be obtained by solving the equation $\partial H/\partial\dot{\mu} = 0$. Taking into account Eq. (2.11) one finds after some algebra:

$$\dot{\mu}_{opt} = \sqrt{-\psi_1(z-1)} - 1 = \sqrt{C'} - 1 = ct \quad (2.15)$$

Here Eq. (2.14) was also used. From Eq. (2.15) one learns that the optimum mass flow rate is constant in time. Use of Eq. (2.15) allows to solve Eq. (2.10) for the unknown function $z(\omega)$. In addition, taking into account the boundary conditions (2.10) yields the integration constant C', which in turn yields the optimum mass flow rate from Eq. (2.15). The result is:

$$\dot{\mu}_{\rm opt} = \frac{\ln\frac{z_{\rm init}-1}{z_{\rm fin}-1}}{\tau_{\rm c} - \ln\frac{z_{\rm init}-1}{z_{\rm fin}-1}} \tag{2.16}$$

The optimum dimensionless temperature time distributions are found from Eqs. (2.8) and (2.9). They are:

$$z_{\rm opt}(\omega) = 1 + (z_{\rm init} - 1)\exp\left(-\frac{\tau_{\rm c}\dot{\mu}_{\rm opt}}{1+\dot{\mu}_{\rm opt}}\omega\right)$$

$$y_{\rm opt}(\omega) = \frac{z_{\rm opt}(\omega) + \dot{\mu}_{\rm opt}}{1+\dot{\mu}_{\rm opt}} \tag{2.17, 18}$$

The minimum dimensionless cooling fluid mass is easily found from Eqs. (2.11) and (2.16):

$$\mu_{\rm min} = \frac{\tau_{\rm c}\ln\frac{z_{\rm init}-1}{z_{\rm fin}-1}}{\tau_{\rm c} - \ln\frac{z_{\rm init}-1}{z_{\rm fin}-1}} \tag{2.19}$$

A few comments follow. The minimum cooling fluid mass given by Eq. (2.19) increases by increasing the cooling time interval $\tau_{\rm c}$. However, this increasing is not linear in $\tau_{\rm c}$ as one might expect from a cooling process with constant mass flow rate. The reason is as follows. The cooling fluid mass is, indeed, linear in the mass flow rate but the last quantity is not linear in $\tau_{\rm c}$, as Eq. (2.16) shows. For an infinitely long cooling process the mass flow rate vanishes but the cooling fluid mass is still finite:

$$\lim_{\tau_{\rm c}\to\infty}\dot{\mu}_{\rm opt} = 0 \qquad \lim_{\tau_{\rm c}\to\infty}\mu = \ln\frac{z_{\rm init}-1}{z_{\rm fin}-1} \tag{2.20, 21}$$

Here Eqs. (2.16) and (2.19) were used again. The result Eq. (2.21) was previously obtained in [1] by using a different approach.

There is a minimum time interval $\tau'_{\rm c,min}$ needed by the optimum cooling process. It is obtained from Eq. (2.16)

$$\tau'_{\rm c,min} = \ln\frac{z_{\rm init}-1}{z_{\rm fin}-1} \tag{2.22}$$

One needs $\tau_{\rm c} > \tau_{\rm c,\ min}$ in order to have a finite, positive, optimum mass flow rate.

3. Forced cooling processes with minimization of dissipation measures

3.1. Dissipation measures

A wildly used measure of dissipation is entropy generation. The entropy generation rate associated to the heat flux $Q_{\rm b\to f}$ transferred from the cooled body to the cooling fluid is denoted $\dot{S}_{\rm gen}$ and is given by:

$$\dot{S}_{\rm gen} = Q_{\rm b\to f}\left(\frac{1}{T_{\rm f,out}} - \frac{1}{T}\right) \tag{3.1}$$

The entropy generation $S_{\rm gen}$ is obtained by integrating Eq. (3.1) for the duration $t_{\rm c}$ of the cooling process:

$$S_{\rm gen} = \int_0^{t_{\rm c}} \dot{S}_{\rm gen}\,{\rm d}t \tag{3.2}$$

Two additional dissipation measures will be considered now. They have in common the notion of lost (available) work. The analysis is more involved than in case of entropy generation because at least one additional system (the work reservoir) must be considered. This increases considerably the number of possible cases and two classes of cases were described in [20]. In the first class (say A), the meta-system (or the universe) consists of three systems (to be more specific, these systems are here. the cooled body, the cooling fluid and the work reservoir). In the second class (B), an environment is added to the previous three systems. What of these ways of defining the lost available work is to be used depends of course on the practical application. A number of examples were presented in [20] but due to the lack of space just two of them (denoted (a) and (b), respectively) will be used in the following. They belong to the class A above.

(a) One could ask what is the lost work rate (say $\dot{W}_{\rm l(a)}$) in case the body looses the heat flux $Q_{\rm b\to f}$. This implies using a reversible refrigeration engine whose coefficient of performance is COP = $T_{\rm f,out}/(T-T_{\rm f,out})$. Then, use of Eq. (3.1) yields:

$$\dot{W}_{\rm l(a)} = Q_{\rm b\to f}/{\rm COP} = T\dot{S}_{\rm gen} \tag{3.3}$$

(b) One could ask what is the work rate (say $\dot{W}_{\rm l(b)}$) to be lost in case of heating the cooling fluid by a heat flux $Q_{\rm b\to f}$. This implies using a reversible heat pump whose coefficient of performance is COP = $T/(T-T_{\rm f,out})$. By using Eq. (3.1) one finds:

$$\dot{W}_{\rm l(b)} = Q_{\rm b\to f}/{\rm COP} = T_{\rm f,out}\dot{S}_{\rm gen} \tag{3.4}$$

In practice, choosing between cases (a) and (b) above depends on the usage of the energy stored by the cooling fluid, after the cooling process is completed. The Eqs. (3.3) and (3.4) connect the rates of lost available work, $\dot{W}_{\rm l(a)}$ and $\dot{W}_{\rm l(b)}$, respectively, with the entropy generation rate $\dot{S}_{\rm gen}$. In both equations the temperature multiplying $\dot{S}_{\rm gen}$ is generally a time dependent quantity. As a consequence, the minimum of the lost available work does not normally coincide with the minimum of the entropy generation.

The lost available work for both above cases, $W_{\rm l(a)}$ and $W_{\rm l(b)}$, respectively, is obtained by integrating Eqs. (3.3) and (3.4) for the whole cooling process:

$$W_{\rm l(a)} = \int_0^{t_{\rm c}} \dot{W}_{\rm l(a)}\,{\rm d}t \quad W_{\rm l(b)} = \int_0^{t_{\rm c}} \dot{W}_{\rm l(b)}\,{\rm d}t \tag{3.5, 6}$$

Note that the absolute value of the lost available work is considered here.

Table 1
Dimensionless dissipation measure rates and dissipation measures. Notations Eqs. (2.5) and (2.6) were used

	a. Dimensionless dissipation measure rate	Equations used
a1	$\tilde{\dot{S}}_{\text{gen}} \equiv \frac{\dot{S}_{\text{gen}}}{\text{UA}} = \frac{\dot{\mu}^2(z-1)^2}{z(1+\dot{\mu})(z+\dot{\mu})}$	(2.1), (3.1)
a2	$\tilde{\dot{W}}_{\text{l(a)}} \equiv \frac{\dot{W}_{\text{l(a)}}}{\text{UA}T_0} = \frac{\dot{\mu}^2(z-1)^2}{(1+\dot{\mu})(z+\dot{\mu})}$	(2.1), (3.1), (3.3)
a3	$\tilde{\dot{W}}_{\text{l(b)}} \equiv \frac{\dot{W}_{\text{l(b)}}}{\text{UA}T_0} = \frac{\dot{\mu}^2(z-1)^2}{(1+\dot{\mu})z^2}$	(2.1), (3.1), (3.4)
	b. Dimensionless dissipation measure	Equations used
b1	$\tilde{S}_{\text{gen}} \equiv \frac{S_{\text{gen}}}{MC} = \tau_{\text{c}} \int_0^1 \frac{\dot{\mu}^2(z-1)^2}{z(1+\dot{\mu})(z+\dot{\mu})} \text{d}\omega$	(2.1), (3.1), (3.2)
b2	$\widetilde{W}_{\text{l(a)}} \equiv \frac{W_{\text{l(a)}}}{MCT_0} = \tau_{\text{c}} \int_0^1 \frac{\dot{\mu}^2(z-1)^2}{(1+\dot{\mu})(z+\dot{\mu})} \text{d}\omega$	(2.1), (3.1), (3.5)
b3	$\widetilde{W}_{\text{l(b)}} \equiv \frac{W_{\text{l(b)}}}{MCT_0} = \tau_{\text{c}} \int_0^1 \frac{\dot{\mu}^2(z-1)^2}{(1+\dot{\mu})z^2} \text{d}\omega$	(2.1), (3.1), (3.6)

Appropriate dimensionless quantities for the dissipation measure rates $\dot{S}_{\text{gen}}$, $\dot{W}_{\text{l(a)}}$ and $\dot{W}_{\text{l(b)}}$, and for the time integrated quantities S_{gen}, $W_{\text{l(a)}}$ and $W_{\text{l(b)}}$ are defined in Table 1, by using the notation Eqs. (2.5) and (2.6).

3.2. Minimization of dissipation measures

Three different objective functions are considered here. They are the dimensionless dissipation measures $\tilde{S}_{\text{gen}}$, $\widetilde{W}_{\text{l(a)}}$ and $\widetilde{W}_{\text{l(b)}}$ defined in Table 1b. Details of the optimization procedure are given now for the only case which allows analytical solutions.

The optimization problem consists in the minimization of the objective function $\widetilde{W}_{\text{l(b)}}$ given by equation in Table 1 b3, taking into account the constraint Eq. (2.8) and the boundary conditions Eq. (2.10). Again, the dimensionless mass flow rate $\dot{\mu}$ is the control function while $\psi_0(\omega)$ and $\psi_1(\omega)$ are the adjunct functions. The Hamiltonian H is defined as follows:

$$H \equiv \psi_0 \left[\frac{\tau_{\text{c}} \dot{\mu}^2 (z-1)^2}{(1+\dot{\mu})^2 z} \right] + \psi_1 \left[-\frac{\tau_{\text{c}} \dot{\mu}}{1+\dot{\mu}} (z-1) \right] \tag{3.7}$$

Again, one can choose $\psi_0 = -1$ and the adjunct function ψ_1 obeys the equation $\text{d}\psi_1/\text{d}\omega = -\partial H/\text{d}z$ i.e.:

$$\frac{\text{d}\psi_1}{\text{d}\omega} = -\frac{\tau_{\text{c}}}{4} \psi_1^2 \tag{3.8}$$

Use of Eqs. (2.8) and (3.8) yields:

$$\psi_1 = -\frac{C''}{\sqrt{z}} \tag{3.9}$$

where $C'' > 0$ is an integration constant. The optimal control function $\mu_{\text{opt}}(\omega)$ is obtained by solving the equation $\partial H/\partial \dot{\mu} = 0$. Taking into account Eq. (3.7), one finds after some algebra:

$$\dot{\mu}_{\text{opt}} = -\frac{\psi_1 z}{2(z-1) + \psi_1 z} = \frac{C'' \sqrt{z}}{2(z-1) - C'' \sqrt{z}} \tag{3.10}$$

Here (3.9) was also used. The differential Eq. (2.8) is solved by using Eq. (3.10) and the boundary conditions (2.10). One finds:

$$z(\omega) = \left[\sqrt{z_{\text{init}}} - \left(\sqrt{z_{\text{init}}} - \sqrt{z_{\text{fin}}}\right)\omega\right]^2$$
$$C'' = \frac{4\left(\sqrt{z_{\text{init}}} - \sqrt{z_{\text{fin}}}\right)}{\tau_{\text{c}}} \tag{3.11, 12}$$

The dependence of the optimum dimensionless mass flow rate $\dot{\mu}_{\text{opt}}$ and of the adjunct function ψ_1 on the dimensionless time ω can be easily found from Eqs. (3.9)–(3.12) and will be not given explicitly here. However, $\dot{\mu}_{\text{opt}}$ must be a positive finite quantity. Consequently, from (3.10) and (3.12) one finds after some algebra the minimum duration $\tau''_{\text{c. min}}$ of the optimized cooling process:

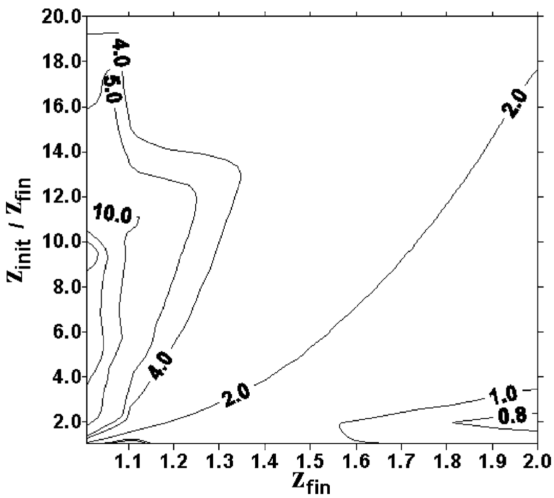

Fig. 2. The ratio $\tau''_{\text{c,min}}/\tau'_{\text{c,min}}$ between the minimum duration of a cooling process minimizing the dimensionless lost available work $\widetilde{W}_{\text{l(b)}}$ and the minimum duration of a cooling process minimizing the dimensionless cooling fluid mass μ, respectively. See Eqs. (2.22) and (3.13) for definitions.

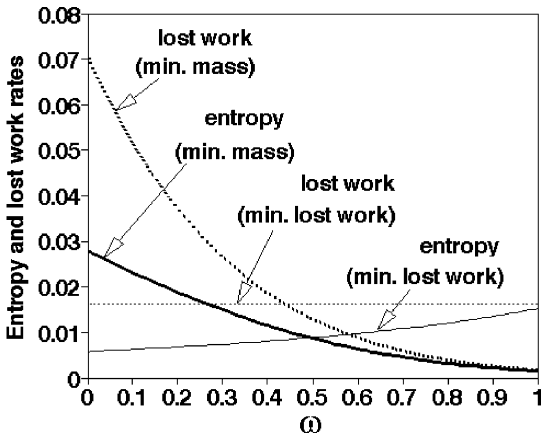

Fig. 3. The dependence of the dimensionless entropy generation rate $\dot{\tilde{S}}_{gen}$ (equation in Table 1 a1) and dimensionless lost available work $\dot{W}_{l(b)}$ (equation in Table 1 a3) on the dimensionless time ω for two optimal control strategies (i.e. minimization of cooling fluid mass and minimization of lost available work $W_{l(b)}$, respectively). The following values were used: $z_{init} = 3$, $z_{fin} = 1.2$, $\tau_c = 10$.

$$\tau''_{c,min} = \frac{2\left(\sqrt{z_{init}} - \sqrt{z_{fin}}\right)}{z_{fin} - 1} \tag{3.13}$$

Here we have also taken into account that the minimum value allowed for $z(\omega)$ is z_{fin}. Fig. 2 shows the ratio $\tau''_{c,min}/\tau'_{c,min}$. Generally, the minimum duration for the cooling process with minimum lost available work considerably exceeds the minimum duration for the cooling processes with minimum mass of cooling fluid. However, for low values of the ratio z_{init}/z_{fin} and high values of z_{fin} the reverse is true.

The difference between the two optimal control strategies (i.e. minimum cooling fluid mass and minimum lost available work) from the point of view of the time variation of the dissipation rates is obvious (Fig. 3). Both the entropy generation rate and the lost available work rate decrease in time in case of the minimization of cooling fluid mass. The strategy of minimizing the lost available work implies a slightly time-increasing entropy generation rate and an almost constant lost work rate.

The minimum dimensionless lost available work $\widetilde{W}_{l(b)}$ is obtained by using equations in Table 1 b3 and Eqs. (3.10)–(3.12). After integration one finds the simple relationship:

$$\widetilde{W}_{l(b),min} = \frac{4}{\tau_c} \tag{3.14}$$

Note that $\widetilde{W}_{l(b),min}$ does not depend on z_{init} or z_{fin}, as the minimum mass of cooling fluid Eq. (2.19) does. Also, in the limit of an infinitely long cooling process (i.e. $\tau_c \to \infty$), $\widetilde{W}_{l(b),min}$ given by (3.14) vanishes, in agreement with well-known results of classical thermodynamics.

The above optimization procedure can be repeated in case of the other dissipation measures, namely $\widetilde{S}_{gen}$ and $\widetilde{W}_{l(a)}$ (see equations in Table 1 b1 and b2). These two cases do no allow analytical solutions. Table 2 summarizes the equations involved. These equations should be solved numerically, together with Eq. (2.8) and the boundary conditions for z (i.e. Eq. (2.10)). Note that no boundary condition is known for the adjunct function ψ_1. This is a rather common situation when optimal control problems are solved. The following procedure was adopted. A trial value (say $\widehat{\psi}_{1,init}$) for the boundary value $\psi_1(\omega = 0)$ was chosen. For that trial boundary value, the Eq. (2.8) and the appropriate equation in Table 2 for the time variation of ψ_1 were solved numerically, starting from $\omega = 0$, where the boundary value for z is known (i.e. $z = z_{init}$). The result obtained for z at $\omega = 1$ (say $\widehat{z}_{fin}$) is compared with the expected boundary value z_{fin} of Eq. (2.10) and the following quantity is computed:

$$F\left(\widehat{\psi}_{init}\right) \equiv \left(z_{fin} - \widehat{z}_{fin}\right)^2 \tag{3.15}$$

$F\left(\widehat{\psi}_{1,init}\right)$ vanishes for the right choice of $\widehat{\psi}_{1,init}$. In case of a significantly large value of $F\left(\widehat{\psi}_{1,init}\right)$, another value

Table 2
Minimization of two dissipation measures. Equations to be solved together with Eq. (2.8)

	a. Minimization of dimensionless entropy generation $\widetilde{S}_{gen}$ (see equation in Table 1 b1)
a1	$\frac{d\psi_1}{d\omega} = -\frac{\tau_c\dot{\mu}}{1+\dot{\mu}}\left[\frac{\dot{\mu}(z\dot{\mu}+\dot{\mu}+2z)(z-1)}{(z+\dot{\mu})^2 z^2} + \psi_1\right]$
a2	$\dot{\mu}_{opt} = \frac{-z(R-1)+z\sqrt{R(R+z-1)}}{R(z+1)-1}$ with $R \equiv -\frac{z-1}{z\psi_1}$
	b. Minimization of dimensionless lost available work $\widetilde{W}_{l(a)}$ (see equation in Table 1 b2)
b1	$\frac{d\psi_1}{d\omega} = -\frac{\tau_c\dot{\mu}}{1+\dot{\mu}}\left[\frac{\dot{\mu}(z+2\dot{\mu}+1)(z-1)}{(z+\dot{\mu})^2} + \psi_1\right]$
b2	$\dot{\mu}_{opt} = \frac{-z(R'-1)+z\sqrt{R'(R'+z-1)}}{R'(z+1)-1}$ with $R' \equiv -\frac{z-1}{\psi_1}$

of $\widehat{\psi}_{1,\mathrm{init}}$ is chosen and the procedure is repeated. In practice, $F\left(\widehat{\psi}_{1,\mathrm{init}}\right)$ was minimized by using the routine FMIN from [21]. Once the appropriate value of $\widehat{\psi}_{1,\mathrm{init}}$ was determined, the Eq. (2.8) and the appropriate differential equation in Table 2 were solved for the optimal paths of z and ψ_1.

Fig. 4 shows the time dependence of the dimensionless temperature z for all the four optimal control strategies envisaged in this paper. The associated optimal cooling fluid mass flow rates $\dot{\mu}_{\mathrm{opt}}$ are presented in Fig. 5. The minimum cooling fluid mass strategy and the minimum entropy generation strategy show a rather non-linear time dependence of z. Interestingly, the associated optimum mass flow rate is constant in time for the first strategy and nearly constant for the second strategy. In the last case a minimum value of $\dot{\mu}_{\mathrm{opt}}$ is however obvious. The two strategies of lost available work minimization imply a slightly nonlinear or even a linear time dependence of z (Fig. 4). This is a consequence of the rather strong non-linear increase in the optimum mass flow rate at the end of the cooling process (Fig. 5).

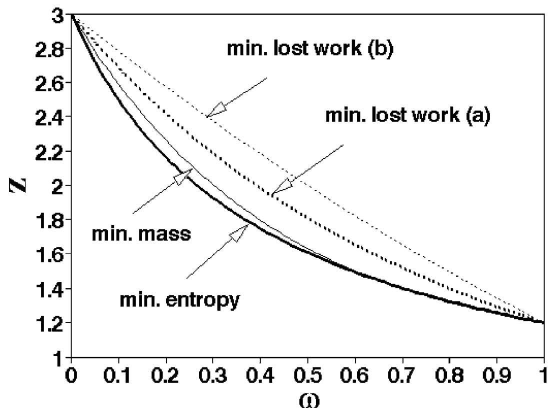

Fig. 4. Dependence of dimensionless temperature z on dimensionless time ω in case of four optimal control strategies: (1) minimum dimensionless cooling fluid mass μ (objective function defined in Eq. (2.11)); (2) minimum dimensionless entropy generation $\tilde{S}_{\mathrm{gen}}$ (objective function defined in equation Table 1 b1); (3) minimum dimensionless lost available work $\widetilde{W}_{\mathrm{l(a)}}$ (objective function defined in equation Table 1 b2); (4) minimum dimensionless lost available work $\widetilde{W}_{\mathrm{l(b)}}$ (objective function defined in equation Table 1 b3). The following values were used: $z_{\mathrm{init}} = 3$, $z_{\mathrm{fin}} = 1.2$, $\tau_{\mathrm{c}} = 10$.

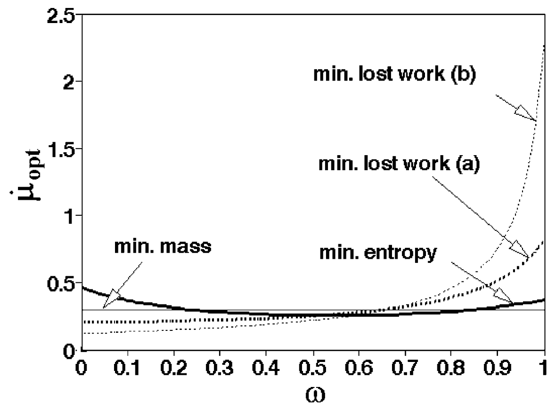

Fig. 5. Dependence of dimensionless cooling fluid mass flow rate $\dot{\mu}_{\mathrm{opt}}$ on dimensionless time ω for the four optimal control strategies of Fig. 4. Again, $z_{\mathrm{init}} = 3$, $z_{\mathrm{fin}} = 1.2$, $\tau_{\mathrm{c}} = 10$.

4. Conclusion

Optimization of forced cooling processes is important for many branches of industry, including cryogenics, electric energy transportation and metallurgy. Optimization techniques based on classical thermodynamics, heat transfer and fluid mechanics are frequently used by designers. More sophisticated methods using Euler–Lagrange equations were also proposed [1]. In this paper the optimization of forced cooling-down processes is treated by using an even more advanced method, namely the optimal control theory.

Various objective functions could be envisaged during optimization. They include the consummated cooling fluid mass and a number of dissipation measures, as for example entropy generation and the lost available work. Four objective functions were shortly presented and minimized in this paper. Using two of the objective functions allows analytical solutions while the other two require numerical procedures. In practice, choosing between these objective functions depends on the particular implementation of the cooling process and on the usage of the energy stored by the cooling fluid, after the cooling process is completed.

The main conclusions of this work are:

(1) The minimum cooling fluid mass is obtained for a constant mass flow rate. The consummated fluid mass increases by increasing the cooling process duration t_{c}. This increasing is not linear in t_{c}, as one might expect from a cooling process with constant mass flow rate.
(2) For an infinitely long cooling process the mass flow rate vanishes but the cooling fluid mass is still finite, in agreement with [1].
(3) There is a minimum time interval needed by the optimum cooling process. It depends on the chosen objective function. The minimum duration for the cooling process with minimum lost available work normally exceeds the minimum duration for the cooling processes with minimum mass of cooling fluid, as one might expect. However, the reverse is true for some particular cases.

(4) The minimum lost available work $\widetilde{W}_{1(b),\min}$ (see Eq. (3.14)) does not depend on the initial and final temperatures of the cooled body but just on the duration of the cooling process.
(5) Those cooling strategies with rather non-linear time dependence of cooled body temperature are based on constant, or nearly constant, optimum mass flow rates of the cooling fluid. The cooling strategies with linear (or slightly non-linear) time dependence of cooled body temperature are based on rather strong non-linear increase in the cooling fluid optimum mass flow rate at the end of the cooling process.

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